Rational Function $f(x) = \dfrac{N(x)}{D(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_2 x^2 + b_1 x + b_0}$

where N(x) and D(x) are polynomials

Numerator N(x) has degree n Denominator D(x) has degree m

The graph of f(x) has vertical asymptotes at the real zeros of the denominator D(x)

The real zeros of denominator D(x) give the values of x that are EXCLUDED from the domain of f(x)

If the denominator has no real zeros, then f(x) does not have any vertical asymptotes

The graph of f(x) has x-intercepts at the real zeros of the numerator N(x)

If the numerator has no real zeros, then f(x) does not have any x-intercepts

The behavior of f(x) as x → – ∞ and as x → ∞ is determined by the behavior of the graph of the ratio of the leading terms of the numerator and denominator $\dfrac{a_n x^n}{b_m x^m} = \dfrac{a_n}{b_m} x^{n-m}$, leading to the rules in the table below

Rules for Determining Asymptotes of f(x) as x → – ∞ and as x → ∞	
The behavior of f(x) as x → – ∞ and as x → ∞ depends on the degree n of N(x)and degree m of D(x)	
n < m	**f(x) has horizontal asymptote y = 0** As x → – ∞ and as x → ∞, y → 0
n = m	**f(x) has horizontal asymptote y =** $\dfrac{a_n}{b_m}$ (ratio of the leading coefficients) As x → – ∞ and as x → ∞, y → $\dfrac{a_n}{b_m}$
n>m: *2 cases*	**f(x) does not have a <u>horizontal</u> asymptote as x → – ∞ and as x → ∞**
n > m and **n = m + 1**	This is a special case of n > m: f(x) has slant asymptote: a line with slope $\dfrac{a_n}{b_m}$. **The equation of the slant asymptote is** the linear equation that is obtained as **the quotient obtained when N(x) is divided by D(x)** (*ignore any remainder*)
n > m and **n > m+1**	**No horizontal or slant linear asymptotes as x → – ∞ and as x → ∞** As x → – ∞ and as x → ∞, f(x) follows behavior of graph y = $\dfrac{a_n x^n}{b_m x^m} = \dfrac{a_n}{b_m} x^{n-m}$

Holes: If (x–*a*) is a linear factor of both the numerator N(x) and the denominator D(x), and if linear factor (x–*a*) has the same multiplicity (exponent) in both numerator N(x) and denominator D(x),
you can divide that factor out to simplify both numerator and denominator,
BUT YOU MUST eliminate the value x = *a* from the domain of the function.
This creates a HOLE in the graph, rather than an x-intercept or a vertical asymptote.

Expanded and Factored Form: To analyze and sketch the graph of a rational function, it is helpful to see the numerator and denominator in factored form as we3ll as in expanded (standard) form.

Definition of vertical asymptote:

The line x = a is a vertical asymptote if as x → a either from the right or the left, the values of f(x) → ∞ or the values of f(x) → – ∞

Definition of horizontal asymptote:

The line y = c is a horizontal asymptote if as x → – ∞ or as x → ∞, f(x) → c

To draw the graph of a rational function: (see page 187 in text)

1. Write f in expanded form and also in factored form (if possible)
2. Plot the y intercept by identifying f(0)
3. Find all real zeros of the numerator (if any) by solving N(x) = 0.
   Then plot *and label* the corresponding x intercepts
4. Find all real zeros of the denominator (if any) by solving D(x) = 0.
   Then sketch *and label* the corresponding vertical asymptotes
5. Find, sketch, *and label* the horizontal (or slant) asymptotes, if any using the rules above.
6. Plot at least one point between and one point beyond each e intercept and vertical asymptote (*and use more points if you need more guidance to complete the graph!*) Make a table that lists all the appropriate intervals, and then select a test point in each interval to evaluate in order to determine points on the graph of the function.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes, using the asymptotes, x intercepts, and plotted points, as guides.

We will do some of the following examples in class; examples C and F are not in the text

A. $f(x)=\dfrac{2x^2}{x^2-1}=\dfrac{2x^2}{(x-1)(x+1)}$ (Example 2a p.186)

B. $f(x)=\dfrac{x}{x^2-x-2}=\dfrac{x}{(x+1)(x-2)}$ (Example 5 p.189)

C. $f(x)=\dfrac{4}{x^2+1}$ (*answer on the next page*)

D. $f(x)=\dfrac{x^2-9}{x^2-2x-3}=\dfrac{(x-3)(x+3)}{(x-3)(x+1)}=\dfrac{x+3}{x+1}, x\neq 3$ (Example 6 p. 189)

E. $f(x)=\dfrac{x^2-x-2}{x-1}=\dfrac{(x-2)(x+1)}{(x-1)}$ (Example 7 p.190)

F. $f(x)=\dfrac{-3x^2-1}{x^2+1}$ (*answer on the next page*)

WHY DO YOU NEED TO LEARN TO DO THIS BY HAND INSTEAD OF RELYING ON YOUR CALCULATOR TO DO THE WORK FOR YOU?

I can aks you to graph a function that has asymptotes and/or intercepts that do not show in the standard graphing window. You would need to understand how to draw the graph from the equation, because I can give you a function that you will not be able to graph adequately using your calculator unless you already have a good enough picture of it in your head to properly adjust and select the proper graphing window or windows.

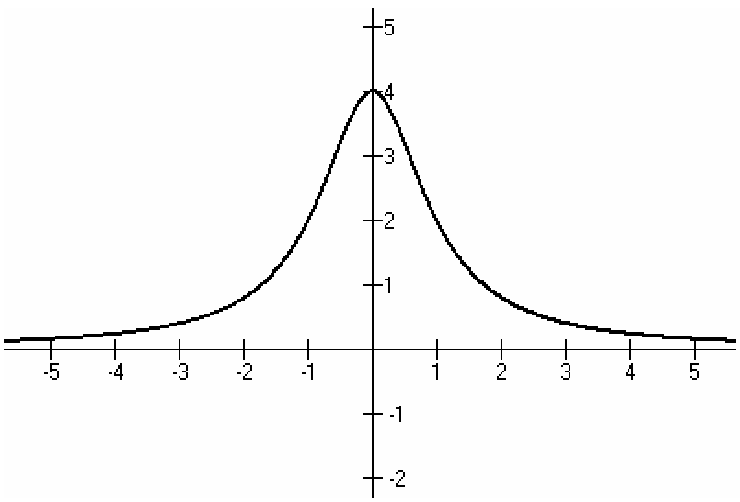
The graphing calculator may be difficult to use to see the graph of a rational function.

For some rational functions it is not possible to view and understand all of the important behavior of the function in a single graphing window because of the scale of the various parts of the graph. For f(x) = $(36x^2 - 9)/(x^2 - 144)$. Different windows are needed to clearly see the x intercepts and to clearly see the asymptotes; the standard viewing window does appear to show any of the important behavior of this function accurately.

Try It: Graph f(x) = $(36x^2 - 9)/(x^2 - 144)$. First use the standard graphing window, which gives a very misleading view of this graph. Then try to accurately see all features of the graph of f(x). You will need to zoom in from the standard window in order get enough detail to accurately view the x intercepts. You will need to zoom out from the standard window once or twice, depending on your calculator model and settings, in order to see all the horizontal and vertical asymptotes.

Answer to C. $f(x) = \dfrac{4}{x^2 + 1}$

y intercept (0,4); no x intercepts,
no vertical asymptotes
horizontal asymptote y = 0
since there are no x intercepts and y intercept is above the x axis, the entire graph must lie above the x axis.
since there are few guides to graphing the function, plot several points for example x = –3, x = –1, x = 1, x = 3



Answer to F. $f(x) = \dfrac{-3x^4 - 1}{x^4 + 1}$

y intercept (0,–1); no x intercepts,
no vertical asymptotes
horizontal asymptote y = –3
since there are no x intercepts and y intercept is below the x axis, the graph must lie entirely below the x axis
since there are few guides to graphing the function, plot several points for example x = −2, −1, 1, 2